

Quantitative Literacy: Thinking Between the Lines

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Chapter 8: Voting and Social Choice

Chapter 8: Voting and Social Choice

Lesson Plan

- ▶ Measuring voting power: Does my vote count?
- ▶ Voting systems: How do we choose a winner?
- ▶ Fair division: What is a fair share?
- ▶ Apportionment: Am I represented?

Chapter 8 Voting and Social Change

8.1 Measuring voting power: Does my vote count?

Learning Objectives:

- ▶ Understand the relationship between a voter's power and the number of votes he or she has
- ▶ Measure true voting power:
 - ▶ Using the Banzhaf index
 - ▶ Using the Shapley-Shubik index

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8.1 Measuring voting power: Does my vote count?

- ▶ **Voting Coalition:** A group of voters who vote the same way
- ▶ **Winning Coalition:** A set of voters with enough votes to determine the outcome of an election, otherwise it is a **Losing Coalition**
- ▶ **Quota:** The number of votes necessary to win the election, in a voting system
- ▶ **Example:** Suppose there are three delegates to a county convention: Abe has 4 votes from his precinct, Ben has 3 votes, and Condi has 1 vote. A simple majority of the votes wins.
 1. What is the quota?
 2. Make a table listing all of the coalitions of voters. Designate which of them are winning coalitions.

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► **Solution:**

1. There are 8 votes, so the quota for simple majority is 5 votes.
2. The following table shows all possible coalitions and votes each coalition controls. The last column indicates whether the coalition controls a majority of the votes and is a winning coalition.

Number of Votes			Total Votes	Winning Coalition?
4	3	1		
Abe	Ben	Condi	8	Yes
Abe	Ben		7	Yes
Abe		Condi	5	Yes
Abe			4	No
	Ben	Condi	4	No
	Ben		3	No
		Condi	1	No

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- ▶ **Critical voter:** A member of a winning coalition is a critical voter if the coalition becomes a losing coalition when that voter is removed.
- ▶ **Example:** The table below contains information from the three winning coalitions in the previous example.

Number of Votes			Total Votes	Winning
4	3	1		
Abe	Ben	Condi	8	Yes
Abe	Ben		7	Yes
Abe		Condi	5	Yes

- ▶ In the first coalition: Abe is the critical voter.
- ▶ In the second coalition: both are critical voters.
- ▶ In the third coalition: both are critical voters.

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- ▶ The previous information can be summarized in the following *coalition table*:

TABLE 8.1 Coalition Table

Number of votes			Total votes	Winning coalition?	Critical voters
4	3	1			
Abe	Ben	Condi	8	Yes	Abe
Abe	Ben		7	Yes	Abe, Ben
Abe		Condi	5	Yes	Abe, Condi
Abe			4	No	Not applicable
	Ben	Condi	4	No	Not applicable
	Ben		3	No	Not applicable
		Condi	1	No	Not applicable

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Winning Coalition and Critical Voters

A set of voters with enough voters to determine the outcome of an election is a *winning coalition*. A voter in a winning coalition is *critical* for that coalition if the coalition is no longer a winning one when that voter is removed. We can summarize the essential information about coalitions in a *coalition table*.

Counting coalitions: Number of coalitions

For n voters, there are $2^n - 1$ possible coalitions (each of which includes at least one voter).

- ▶ **Example:** If there are 7 voters, there are 2^7 possibilities for voters to be in or not in a coalition. This includes the possibility of all the voters not being in any coalition, so there are $2^7 - 1$ possibilities.

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- ▶ **Banzhaf power index:** The number of times a voter is critical in a winning coalition divided by the total number of instances in which any voter is critical. Expressed as a fraction or percentage.
- ▶ **Example:** Use coalition Table 8.1 to compute the Banzhaf index for each county convention delegate described in the previous examples.
- ▶ **Solution:** From the coalition table: overall there were 5 instances in which any voter was critical. So the Banzhaf power index of a voter is the number of times that voter is critical divided by 5.
 - ▶ Abe was critical 3 of the 5 times, so he has a Banzhaf power index of $3/5$ or 60%.
 - ▶ Ben was critical 1 of the 5 times, so he has a Banzhaf power index of $1/5$ or 20%.
 - ▶ Condi was critical 1 of the 5 times. So she has a Banzhaf power index of $1/5$ or 20%.

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- ▶ **Example:** At the 1988 Democratic National Convention, Michael Dukakis's delegates had 1401 votes, Jesse Jackson's had 1218 votes, Al Gore's had 325 votes, and Bruce Babbitt's had 197 votes. It is common that candidates form coalitions. A simple majority is required to win the nomination.
 1. What is the quota?
 2. Determine the winning coalitions.
 3. Determine the critical voters (candidates) in each winning coalition.
 4. Determine the Banzhaf index of each candidate.

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► **Solution:**

1. The total number of votes is 3141, half that is 1570.5, so a simple majority is at least 1571 votes, and that is the quota.



Presidential candidate Jesse Jackson addressing the 1988 Democratic Convention.

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► **Solution (cont.):**

- 2. The following table lists the $2^4 - 1 = 15$ positive coalitions

Votes				Total votes	Winning coalition?
1401	1218	325	197		
D	J	G	B	3141	Yes
D	J	G		2944	Yes
D	J		B	2816	Yes
D	J			2619	Yes
D		G	B	1923	Yes
D		G		1726	Yes
D			B	1598	Yes
				1401	No
	J	G	B	1740	Yes
	J	G		1543	No
	J		B	1415	No
	J			1218	No
		G	B	522	No
		G		325	No
			B	197	No

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▶ **Solution (cont.):**

3. The accompanying table lists the winning coalitions only, along with the critical voters in each case.

Winning Coalitions Only					
Votes					
1401	1218	325	197	Total votes	Critical voters
D	J	G	B	3141	None
D	J	G		2944	D
D	J		B	2816	D
D	J			2619	D, J
D		G	B	1923	D
D		G		1726	D, G
D			B	1598	D, B
	J	G	B	1740	J, G, B

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▶ **Solution (cont.):**

4. There are 12 instances in which a voter or candidate is critical:

▶ Dukakis is critical 6 of the 12 times, so his Banzhaf index is

$$\frac{6}{12} = \frac{1}{2} \text{ or } 50\%$$

▶ Jackson is critical 2 of the 12 times, so his Banzhaf index is

$$\frac{2}{12} = \frac{1}{6} \text{ or } 16.67\%$$

▶ Gore is critical 2 of the 12 times, so his Banzhaf index is

$$\frac{2}{12} = \frac{1}{6} \text{ or } 16.67\%$$

▶ Babbitt is critical 2 of the 12 times, so his Banzhaf index is

$$\frac{2}{12} = \frac{1}{6} \text{ or } 16.67\%$$



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- ▶ **Benzhaf Power Index:** A voter's *Benzhaf index* is the number of times that voter is critical in some winning coalition divided by the total number of instances in which any voter is critical. The index is expressed as a fraction or as a percentage.
- ▶ **Swing voter:** Supposing the voters vote in order and their votes are added as they vote, **the swing voter** is the voter whose votes make the total meet the quota and thus decide the outcome. Which is the swing voter depends on the order the votes are cast.

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- ▶ **Example:** Members of the European Union have votes on the Council determined roughly by a country's population but progressively weighted in favor of smaller countries. Ireland has 7 votes, Cyprus has 4 votes, and Malta has 3 votes. Supposing a simple majority wins, make a table with all the permutations of voters and the swing voter in each case.
- ▶ **Solution:** There are 14 votes total so a simple majority is 8. So the quota is 8. There are $n!$ different permutations of n objects, where

$$n! = n \times (n - 1) \times \cdots \times 1$$

So the 3 objects have $3! = 3 \times 2 \times 1 = 6$ permutations. The following table lists the permutations and swing voter in each case.

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► **Solution (cont.):**

	Order of Voters			Swing Voter
Ireland (7)	Cyprus (4)	Malta (3)		Cyprus
Ireland (7)	Malta (3)	Cyprus (4)		Malta
Cyprus (4)	Ireland (7)	Malta (3)		Ireland
Cyprus (4)	Malta (3)	Ireland (7)		Ireland
Malta (3)	Ireland (7)	Cyprus (4)		Ireland
Malta (3)	Cyprus (4)	Ireland (7)		Ireland

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- ▶ **The Shapley-Shubik power index:** is calculated as the fraction (or percentage) of all permutations of the voters in which that voter is the swing.
- ▶ **Example:** Compute the Shapley-Shubik power index for the committee of Ireland, Cyprus, and Malta from the previous example.
- ▶ **Solution:** There are six permutations of the voters. Ireland is the swing in 4 of the 6 cases so the index for Ireland is $4/6 = 2/3$ or about 66.67%.
 - ▶ Cyprus is the swing in 1 of the 6 cases so its index is $1/6$ or 16.67%.
 - ▶ Malta also is the swing in 1 of the 6 cases so its index is $1/6$ or 16.67%.

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- ▶ **Example:** In the 2004 election, the votes from Florida, Michigan, Ohio, and Pennsylvania were important in determining the outcome. Florida had 27 electoral votes, Michigan had 17, Ohio had 20, and Pennsylvania had 21. Assume a majority from only these four states would determine the election. In this case the quota is 43.
 1. How many permutations of these four states are there?
 2. Make a table listing each case and its swing voter.
 3. Find the Shapley-Shubik index for each state.
- ▶ **Solution:**
 1. The number of permutations of four items is
$$4! = 4 \times 3 \times 2 \times 1 = 24$$
 2. The following table shows the different permutations and the swing vote state in each case.

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► Solution (cont.):

Voters				Swing
FL (27)	MI (17)	OH (20)	PA (21)	MI
FL (27)	MI (17)	PA (21)	OH (20)	MI
FL (27)	MI (20)	MI (17)	PA (21)	OH
FL (27)	MI (20)	PA (21)	MI (17)	OH
FL (27)	MI (21)	MI (17)	OH (20)	PA
FL (27)	MI (21)	OH (20)	MI (17)	PA
MI (17)	FL (27)	OH (20)	PA (21)	FL
MI (17)	FL (27)	PA (21)	OH (20)	FL
MI (17)	OH (20)	FL (27)	PA (21)	FL
MI (17)	OH (20)	PA (21)	FL (27)	PA
MI (17)	PA (21)	FL (27)	OH (20)	FL
MI (17)	PA (21)	OH (20)	FL (27)	OH
OH (20)	FL (27)	MI (17)	PA (21)	FL
OH (20)	FL (27)	PA (21)	MI (17)	FL
OH (20)	MI (17)	FL (27)	PA (21)	FL
OH (20)	MI (17)	PA (21)	FL (27)	PA
OH (20)	PA (21)	FL (27)	MI (17)	FL
OH (20)	PA (21)	MI (17)	FL (27)	MI
PA (21)	FL (27)	MI (17)	OH (20)	FI
PA (21)	FL (27)	OH (20)	MI (17)	FL
PA (21)	MI (17)	FL (27)	OH (20)	FL
PA (21)	MI (17)	OH (20)	FL (27)	OH
PA (21)	OH (20)	FL (27)	MI (17)	FL
PA (21)	OH (20)	MI (17)	FL (27)	MI

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▶ **Solution (cont.):**

3. Florida has the swing in 12 cases, Michigan has the swing in 4 cases, Ohio has the swing in 4 cases, and Pennsylvania has the swing in 4 cases.

So the index for Florida is $12/24 = 1/2$ or 50%,

Michigan: $4/24 = 1/6$ or about 16.67%,

Ohio: $4/24 = 1/6$ or about 16.67%,

Pennsylvania: $4/24 = 1/6$ or 16.67%.

The Shapley-Shubik Index of a given voter is calculated as the fraction of all permutations of the voters in which that voter is the swing.

Chapter 8 Voting and Social Change: **Chapter Summary**

- ▶ **Voting power:** does your vote count
 - ▶ coalitions, quotas, swing voters
 - ▶ Banzhaf power index
 - ▶ Stanley-Shubik power index

- ▶ **Voting systems:** choosing a winner
 - ▶ plurality
 - ▶ top-two runoff
 - ▶ elimination runoff
 - ▶ Borda count
 - ▶ Condorcet winner

Chapter 8 Voting and Social Change: **Chapter Summary**

▶ **Fair division:**

- ▶ Methods: divide and choose
adjusted winner procedure

▶ **Apportionment:**

- ▶ Hamilton's method
- ▶ Jefferson's method
- ▶ Adjusted divisor methods
- ▶ Huntington-Hill method

